

VELOCITY REVERSAL CRITERION OF A BODY IMMERSED IN A SEA OF PARTICLES

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ABSTRACT. We consider a rigid body colliding with a continuum of particles. We assume that the body is moving at a velocity close to an equilibrium velocity V_∞ and that the particles colliding with the body reflect diffusely, that is, probabilistically with some probability distribution K . We find a condition that is sufficient and almost necessary that the collective force of the colliding particles reverses the relative velocity $V(t)$ of the body, that is, changes the sign of $V(t) - V_\infty$, before the body approaches equilibrium. Examples of both reversal and irreversal are given. This is in strong contrast with the pure specular reflection case in which only reversal happens.

1. INTRODUCTION

The problem that we are considering has a free boundary, the location of the body. The other unknown is the configuration of the particles. The particles may collide with the body elastically or diffusely. Boundary interactions in kinetic theory are very poorly understood, even when the boundaries are fixed. Free boundaries are even more difficult. For this reason we have chosen to consider only the *simplest* problem of this type, namely, we assume the particles are identical and are rarefied, that is, do not interact among themselves but only with the body. We assume that the whole system, consisting of the body and the particles, starts out rather close to an equilibrium state.

We consider classical particles that are extremely numerous. While one could consider modeling them as a fluid, we instead model them as a continuum like in kinetic (Boltzmann, Vlasov) theory [8, 10, 11] but without any self-interaction. Our focus is on the interaction of the particles with the body at its boundary. In typical physical scenarios this interaction can be quite complicated. For instance, the boundary may be so rough that a particle may reflect from it in an essentially random way. There could even be some kind of physical or chemical reaction between the particle and the molecules of the body.

The present paper is a sequel to [7] and is also highly motivated by the series of papers [4, 3, 1]. In all four papers the initial velocity $V(0)$ of the body is close to its terminal (equilibrium) velocity $V_\infty > 0$. In [4, 1, 7] the body approaches its equilibrium velocity in such a way that $V(t) < V_\infty$ for all time t . On the other hand, in the paper [3] the body's initial velocity $V(0)$ satisfies $V(0) > V_\infty$ and after a certain time its velocity switches to $V(t) < V_\infty$ before approaching its equilibrium $V(t) \rightarrow V_\infty$ as $t \rightarrow \infty$. In [3] all the particles reflect elastically (specularly).

The purpose of the present paper is to analyze the effect of inelastic (diffusive) collisions given by a probability distribution K and to determine conditions on K so that the velocity of the body reverses or does not reverse, that is, $V(t) - V_\infty$ does or does not change sign. We discover that there are diffusive collision laws that lead to reversal and others that lead to irreversal, no matter what V_∞ and $V(0)$ are, so long as they are close together. These laws are almost exact opposites of each other. In particular, the existence of an irreversal case for $0 < V_\infty < V(0)$ is in direct contrast to the purely specular collision case in [3] where only reversal takes place.

Moreover, in the present paper we prove that, regardless of whether the velocity is reversed or not, the equilibrium is ultimately approached at the same polynomial rate as in [7]. This rate is $O(t^{-d-p})$ in d spatial dimensions where p could take any value in $(0, 2]$, depending on the specific law of reflection given by K . Though purely diffuse collisions with a Gaussian kernel were considered in [1] and both diffuse and elastic collisions were considered in [7], in both of those papers there was no reversal of the

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velocity. Some discussion of the physical motivation of this type of problem can be found in [7] and the other cited references. Some closely related investigations are [2, 5, 6, 9, 12].

To be specific, here we consider the following problem. The body is a cylinder $\Omega(t) \subset \mathbb{R}^d$. We write $\mathbf{x} = (x, x_\perp)$, $x_\perp \in \mathbb{R}^{d-1}$. The cylinder is parallel to the x -axis and the body is constrained to move only in the x direction with velocity $V(t)$. There may be a constant horizontal force $E \geq 0$ acting on the body, as well as the horizontal force $F(t)$ due to all the colliding particles at time t . Thus

$$\frac{dX}{dt} = V(t), \quad \frac{dV}{dt} = E - F(t),$$

If $E = 0$, then the body is at rest in equilibrium ($V_\infty = 0$), while if $E \neq 0$, then $V_\infty \neq 0$ is given by $F_0(V_\infty) = E$, where $F_0(V)$ is the fictitious force in case no particle collides more than once (see (3.1) below). In order to avoid confusion in this paper, we shall take $0 \leq V_\infty < V(0)$.

We introduce the following notation. The velocity of a particle is $\mathbf{v} = (v_x, v_\perp)$, where $v_x = \mathbf{v} \cdot \mathbf{i}$ is the horizontal component and $v_\perp \in \mathbb{R}^{d-1}$. The particle distribution, denoted by $f(t, \mathbf{x}, \mathbf{v})$, satisfies $\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = 0$ in $\Omega^c(t)$. We assume the initial velocity $f(0, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{v})$ depends only on \mathbf{v} and is even in v_x . We also denote the densities before and after a collision with the body by $f_\pm(t, \mathbf{x}, \mathbf{v}) = \lim_{\epsilon \rightarrow 0^+} f(t \pm \epsilon, \mathbf{x} \pm \epsilon \mathbf{v}, \mathbf{v})$. The assumed law of reflection at the two ends of the cylinder is

$$f_+(t, \mathbf{x}; \mathbf{v}) = \int_{(u_x - V(t))(v_x - V(t)) \leq 0} K(\mathbf{v} - \mathbf{i}V(t); \mathbf{u} - \mathbf{i}V(t)) f_-(t, \mathbf{x}; \mathbf{u}) d\mathbf{u}, \quad (1.1)$$

where \mathbf{i} is the unit vector in the x -direction. The collision kernel K is assumed to be even and satisfy the conservation of mass condition (1.6) below. Furthermore, K and the initial density f_0 are assumed to satisfy Assumptions A1-A4 in Section 2. Among these conditions are $f_0(\mathbf{v}) = a_0(v_x)b(v_\perp)$ and

$$K(\mathbf{v}, \mathbf{u}) = k(v_x, u_x)b(v_\perp), \quad c|u_x|^p \leq \int_{v_x \geq 0} v_x^2 k(v_x, u_x) dv_x \leq C|u_x|^p$$

for some constants c, C, p and some function $b(v_\perp)$ where $0 < p \leq 2$.

Theorem 1.1 (Irreversal). *Let K be a collision kernel as above and let f_0 be the initial particle distribution. Let the initial velocity V_0 of the cylinder be slightly larger than V_∞ ; that is, $0 \leq V_\infty < V_0 < V_\infty + \gamma$, where γ is sufficiently small. Assume the Irreversal Criterion*

$$\int_0^\infty k(0, z) a_0(z + V_\infty) dz > a_0(V_\infty). \quad (1.2)$$

(a) *Then there exists at least one solution $(V(t), f(t, x, v))$ of our problem in the following sense. $V \in C^1(\mathbb{R})$ and $f_\pm \in L^\infty$ for $t \in [0, \infty)$, $x \in \partial\Omega(t)$, $v \in \mathbb{R}^3$, where the force $F(t)$ on the cylinder is given by (1.7) below and the pair of functions $f_\pm(t, x, v)$ are (almost everywhere) defined explicitly in terms of $V(t)$ and $f_0(x, v)$.*

(b) *Furthermore, every solution of the problem (in the sense stated above) satisfies the estimates*

$$0 < \gamma e^{-B_0 t} + \frac{c\gamma^{p+1}}{t^{d+p}} \chi\{t \geq t_0 + 1\} < V(t) - V_\infty < \gamma e^{-B_\infty t} + \frac{C\gamma^{p+1}}{(1+t)^{d+p}} \quad (1.3)$$

for $0 < t < \infty$ and for some positive constants c, C, B_0, B_∞ and t_0 that will be specified later. Notice that there is no velocity reversal because $V(t) > V_\infty$ for all $t > 0$.

Theorem 1.2 (Reversal). *Given the same situation as above, except that we now assume the Reversal Criterion*

$$\int_0^\infty k(0, z) a_0(z + V_\infty) dz < a_0(V_\infty). \quad (1.4)$$

(a) *Then there exists at least one solution $(V(t), f(t, x, v))$ of our problem in the sense given in part (a) of the preceding theorem.*

(b) *Furthermore, every solution of the problem (in the sense stated above) satisfies the estimates*

$$\gamma e^{-B_0 t} - \frac{c\gamma^{p+1}}{(1+t)^{d+p}} < V(t) - V_\infty < \gamma e^{-B_\infty t} - \frac{C\gamma^{p+1}}{t^{d+p}} \chi\{t \geq t_0 + 1\} \quad (1.5)$$

for $0 < t < \infty$ and for some positive constants c, C, B_0, B_∞ and t_0 specified later. Notice that the velocity reversal for sufficiently large t is incorporated in this inequality.

The contrasting criteria (1.2) and (1.4) have the following interpretation. The body is initially moving to the right. Letting $u_x = z + V_\infty$, we see that the left side of both inequalities represents the velocity density, after collisions on the left of the body, of the particles with approximately the same velocity as the body. These are the particles that are most likely to collide again later. The particles on the right are less likely to collide with the body again because the body is slowing down initially under the condition that $V_0 > V_\infty$. In the reversal case there will be fewer collisions on the left side compared with the particles that do not collide. Therefore there are fewer future collisions on the left, so that the body tends to move more to the left, and $V(t)$ has more of a chance to cross over from being larger than V_∞ to being smaller. In the irreversal case, there are more such particles, so there are more collisions on the left and the velocity of the body tends to remain larger than V_∞ . Much more subtle, and not studied in this paper, is the case when there is equality in (1.2) and (1.4).

In case $0 < V_0 < V_\infty$, the body initially moves slower than the equilibrium, so the particles on the right now play the critical role. Adapting (1.2) and (1.4) to the right side of the cylinder, the Irreversal Criterion becomes

$$\int_{-\infty}^0 k(0, z) a_0(z + V_\infty) dz > a_0(V_\infty),$$

which was treated in [7], while the Reversal Criterion becomes

$$\int_{-\infty}^0 k(0, z) a_0(z + V_\infty) dz < a_0(V_\infty),$$

which has a very similar proof that we omit. In case $V_\infty = 0$, the criteria on the right and the left coincide due to the evenness of K and f_0 .

We now discuss the basic setup of the problem which is essentially the same as in [7], where more details and derivations may be found. The kernel $k(v_x, u_x)$ is assumed to be nonnegative and even in each variable. Conservation of mass requires that

$$\int_{v_x \geq 0} v_x K(\mathbf{v}, \mathbf{u}) d\mathbf{v} = |u_x|. \quad (1.6)$$

The total horizontal force on the body at time t due to the particles is given in terms of $f_-(t, \mathbf{x}, \mathbf{v})$ as

$$F(t) = \int_{\partial\Omega_R(t) \cup \partial\Omega_L(t)} dS_{\mathbf{x}} \int_{\mathbb{R}^3} d\mathbf{v} \operatorname{sgn}(V(t) - v_x) \ell(\mathbf{v} - \mathbf{i}V(t)) f_-(t, \mathbf{x}, \mathbf{v}), \quad (1.7)$$

where we denote

$$\ell(\mathbf{w}) = w_x^2 + \int_{v_x \geq 0} d\mathbf{v} v_x^2 K(\mathbf{v}, \mathbf{w}). \quad (1.8)$$

On the lateral boundary S of the cylinder we also assume a boundary condition of the form

$$f_+(t, \mathbf{x}; \mathbf{v}) = \int_{\mathbf{u} \cdot \mathbf{n}_{\mathbf{x}} \leq 0} K_S(\mathbf{v}; \mathbf{u}) f_0(\mathbf{u}) d\mathbf{u}.$$

together with the corresponding conservation of mass condition. Then no net force is created on the lateral boundary (see [7, Lemma 2.5]) and the body continues to move horizontally.

In Section 2 we state the precise assumptions on the collision kernel K and the initial particle distribution f_0 , followed by several examples. Example 1 is a Gaussian collision law of both k and a_0 , namely

$$a_0(u_x) = C_1 e^{-\alpha u_x^2}, \quad k(v_x, u_x) = C_2 e^{-\beta v_x^2} |u_x|.$$

If V_∞ either vanishes or is small enough, then it satisfies the Reversal Criterion if $\beta < \alpha$, while it satisfies the Irreversal Criterion provided $\beta > \alpha$. If $V_\infty^2 > \frac{\beta}{2\alpha^2}$, it satisfies the Reversal Criterion. Example 2 has a Gaussian kernel $K(v, u)$ like $\exp(-v_x^2/|u_x|)$, which means that colliding particles with velocities close to that of the body deflect only a little, while colliding particles with very different velocities reflect with a very wide distribution of velocities. Example 3 is more general, permitting an ultimate rate of approach to the body at the rate $O(t^{-d-p})$ for any $p \in (0, 2]$.

Sections 3 and 4 are devoted to the proofs of the irreversal and reversal cases, respectively. In each case a family \mathcal{W} of possible body motions W is introduced. We write the force due to the possible motion W as $F(t) = F_0(t) + R_W(t)$, where $R_W(t)$ is the force due to the collisions occurring before

time t (“precollisions”) if the body were to move with velocity $W(\cdot)$. Then W generates a new possible motion V_W by the equation

$$\frac{dV_W}{dt} = \frac{F_0(V_\infty) - F_0(W(t))}{V_\infty - W(t)} (V_\infty - V_W) - R_W(t)$$

The goal is to prove that the mapping $W \rightarrow V_W$ has a fixed point. The main upper and lower bounds of $R_W(t)$ are stated in Theorem 3.1 for the irreversible case and Theorem 4.1 for the reversal case. Assuming them, Theorems 1.1 and 1.2 follow easily. The proofs of Theorems 3.1 and 4.1 are the core of this paper. The key bound (3.4) in the irreversible case proves that $R_W(t)$ remains negative. On the other hand, $R_W(t)$ stays positive in the reversal case, according to (4.2). We begin the proofs by considering the class \mathcal{W} of possible motions and then prove the required bounds for the particles colliding with the body from the left side, followed by those that collide from the right side. In the reversal case, because $E \geq 0$, the collisions on the right begin to dominate but, after the velocity reversal, eventually those on the right and the left balance each other and the body tends to its equilibrium speed from below.

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2. ASSUMPTIONS AND EXAMPLES OF DIFFUSION KERNELS

We make the following assumptions on the diffusion kernel K , which governs the collisions with the body.

A1. [Structure] Let K and f_0 have the product form

$$\begin{aligned} f_0(\mathbf{v}) &= a_0(v_x)b(v_\perp), \\ K(\mathbf{v}, \mathbf{u}) &= k(v_x, u_x)b(v_\perp), \quad \int b(v_\perp)dv_\perp = 1 \text{ and } b(0) > 0, \end{aligned}$$

with each factor nonnegative and continuous, with f_0 bounded and with a_0 and k even functions in both u_x and v_x .

Notice that, under this assumption, $\ell(\mathbf{w})$ actually depends only on w_x ; that is,

$$\ell(\mathbf{w}) = w_x^2 + \int_{v_x \geq 0} dv_x v_x^2 k(v_x, w_x) = \ell(w_x).$$

Therefore, at any later time f_+ and f_- must take the product form

$$f_+(t, \mathbf{x}; \mathbf{v}) = a_+(t, \mathbf{x}; v_x)b(v_\perp), \quad f_-(t, \mathbf{x}; \mathbf{v}) = a_-(t, \mathbf{x}; v_x)b(v_\perp).$$

We remark that, although collisions occur only in the horizontal direction, the analysis is not entirely one-dimensional; the dimension does come into play as we shall see in the proofs of (1.3) and (1.5).

A2. [Boundedness]

$$\sup_{|u_x| \leq \gamma} \sup_{v_x \in \mathbb{R}} k(v_x, u_x) < \infty.$$

A3. [Power Law] There is a power $0 < p \leq 2$ and there are positive constants C and c such that

$$c|u_x|^p \leq \int_{v_x \geq 0} v_x^2 k(v_x, u_x) dv_x \leq C|u_x|^p$$

for $u_x \in [-\gamma, \gamma]$. We also assume that this integral is an even C^1 function of u_x for $u_x \neq 0$ and is strictly decreasing for $u_x < 0$. Combining A3 and A1, we have

$$c|u_x|^p \leq \ell(u_x) \leq u_x^2 + C|u_x|^p \leq C'|u_x|^p \text{ for } u_x \in [-\gamma, \gamma].$$

A4. [Integrability]

$$k(v_x, z - y - V_\infty) a_0(z) \leq M(z) \quad \text{for } |v_x| < 2\gamma, |y| < \gamma, |z| < \infty,$$

where $M \in L^1(\mathbb{R})$.

2.1. Examples of Collision Kernels. In this section, we give a few examples of collision kernels and initial densities that satisfy the assumptions.

Example 1. *Let*

$$a_0(u_x) = C_1 e^{-\alpha u_x^2}, \quad k(v_x, u_x) = C_2 e^{-\beta v_x^2} |u_x|.$$

The requirement that mass is conserved means that

$$\int_{v_x \geq 0} v_x K(\mathbf{v}, \mathbf{u}) d\mathbf{v} = |u_x|,$$

which reduces to choosing $C_2 = 2\beta$. Assumptions A1-A4 are seen to be easily satisfied with $p = 1$. The Reversal Criterion takes the form

$$2\beta \int_0^\infty z e^{-\alpha(z+V_\infty)^2} dz = 2\beta \int_{V_\infty}^\infty (z - V_\infty) e^{-\alpha z^2} dz < e^{-\alpha V_\infty^2}, \quad (2.1)$$

while the Irreversal Criterion is the opposite (strict) inequality.

Let us first suppose that V_∞ either vanishes or is very small. Then the Reversal Criterion is satisfied provided $\frac{\beta}{\alpha} = 2\beta \int_0^\infty z e^{-\alpha z^2} dz < 1$, or $\beta < \alpha$, while the Irreversal Criterion is satisfied if $\beta > \alpha$. Now α and β may be interpreted as the reciprocals of (normalized) temperatures. So the velocity reverses if the body is hotter than the gas and the speed V_∞ is sufficiently small. The velocity does not reverse if the body has a lower temperature than the gas and the speed V_∞ is sufficiently small. The latter situation could happen for a comet or a space vehicle that actively cools itself during its reentry into the atmosphere.

Next let us consider a fast moving body. We can write (2.1) as

$$\frac{1}{2\alpha} e^{-\alpha V_\infty^2} - \frac{V_\infty \sqrt{\pi}}{2\sqrt{\alpha}} \operatorname{erfc}(\sqrt{\alpha} V_\infty) < \frac{1}{2\beta} e^{-\alpha V_\infty^2}.$$

We can use the asymptotic expansion of the complementary error function

$$\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n}.$$

In fact, two easy integrations by parts yield

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2}\right) + \frac{3}{2\sqrt{\pi}} \int_x^\infty e^{-t^2} \frac{dt}{t^4} > \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2}\right)$$

for any $x > 0$. Thus the Reversal Criterion is satisfied if

$$\frac{1}{4\alpha^2 V_\infty^2} e^{-\alpha V_\infty^2} = \frac{1}{2\alpha} e^{-\alpha V_\infty^2} - \frac{1}{2\alpha} \left(1 - \frac{1}{2\alpha V_\infty^2}\right) e^{-\alpha V_\infty^2} < \frac{1}{2\beta} e^{-\alpha V_\infty^2}.$$

That is, the velocity reverses if $V_\infty^2 > \frac{\beta}{2\alpha^2}$. In particular, if $\alpha = \beta$, the velocity reverses if $V_\infty^2 > \frac{1}{2\alpha}$. This agrees with the numerical calculations of Case 9 in [2]. The case of equal temperatures ($\alpha = \beta$) is motivated by Boltzmann theory [10, 11].

Now consider a fast moving body and look for irreversal. We further expand

$$\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{3}{4x^4}\right) - \frac{15}{4\sqrt{\pi}} \int_x^\infty e^{-t^2} \frac{dt}{t^6}.$$

Dropping the last integral, we deduce that there is no reversal if $\frac{1}{4\alpha^2 V_\infty^2} - \frac{3}{8\alpha^3 V_\infty^4} > \frac{1}{2\beta}$, which can happen if $\beta > 12\alpha$. This again agrees with the numerical data in [2], namely that there is reversal if $\alpha = \beta$.

Example 2. *We now choose*

$$K(\mathbf{v}, \mathbf{u}) = 2e^{-\frac{v_x^2}{|u_x|}} b(v_\perp), \quad f_0(\mathbf{v}) = a_0(v_x) b(v_\perp),$$

as in [7, Example 2]. Mass conservation during collisions forces the coefficient to be 2. We assume $a_0 \in L^1(\mathbb{R})$, and $\int b dv_\perp = 1$. The physical interpretation is that an incoming particle with almost the same velocity as that of the body is likely to be reflected with almost the same velocity, while an incoming particle with velocity quite different from that of the body is reflected according to a wide Gaussian

distribution around $V(t)$. As in [7, Example 2], this collision kernel satisfies A1-A4 with $p = \frac{3}{2}$. The Reversal Criterion then means that

$$\int_{V_\infty}^{\infty} a_0(u) du < \frac{a_0(V_\infty)}{2}, \quad (2.2)$$

while the Irreversal Criterion means the opposite (strict) inequality. A simple instance of reversal is the algebraic decay: $a_0(u) = \frac{1}{u^m}$ for $u \geq 1$, in which case the Reversal Criterion is satisfied so long as $1 \leq V_\infty < \frac{m-1}{2}$ with $m > 4$. A second instance is the Gaussian $a_0(u) = C_1 e^{-\beta u^2}$, for which reversal means

$$\frac{2}{\sqrt{\beta}} \int_{\sqrt{\beta} V_\infty}^{\infty} e^{-z^2} dz < e^{-\beta V_\infty^2}.$$

Using one term in the expansion of erfc with a negative remainder, we see that reversal occurs if $V_\infty < \frac{1}{\beta}$. Similarly, using two terms in the expansion with a positive remainder, we see that irreversal occurs if $\frac{1}{\beta V_\infty} - \frac{1}{2\beta^2 V_\infty^3} > 1$.

Example 3. As in [7, Example 3], we can find a family of kernels that covers a continuous range of p . Given $\beta \in [-1, 3]$, we choose

$$K(\mathbf{v}, \mathbf{u}) = C_2 |u_x|^\beta e^{-v_x^2 |u_x|^{\beta-1}} b(v_\perp), \quad f_0(\mathbf{v}) = a_0(v_x) b(v_\perp).$$

Once again, C_2 is chosen so that mass is conserved during collisions, while a_0 and b are chosen as in Example 2. We then have

$$C_2 |u_x|^\beta \int_0^\infty v_x^2 e^{-v_x^2 |u_x|^{\beta-1}} dv_x = C |u_x|^{\frac{3-\beta}{2}}$$

for some constant C . Thus p runs through $(0, 2]$ as β runs through $[-1, 3]$.

3. PROOF OF THE IRREVERSAL CASE

3.1. Proof assuming the Key Estimate. Theorem 1.1 will be proven by a fixed point technique. We will first define a family \mathcal{W} of possible body motions W . Given a possible motion $W \in \mathcal{W}$, let $F_0(W)$ be the force if each particle were to collide only once and the body were to move with velocity $W(t)$. It is given by putting f_0 in place of f_- in (1.7), that is,

$$\begin{aligned} F_0(W) &= \int_{\partial\Omega_R(t)} dS_{\mathbf{x}} \int_{u_x \leq W(t)} d\mathbf{u} \ell(\mathbf{u} - \mathbf{i}W(t)) f_0(\mathbf{u}) \\ &\quad - \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{u_x \geq W(t)} d\mathbf{u} \ell(\mathbf{u} - \mathbf{i}W(t)) f_0(\mathbf{u}) \\ &= C \left(\int_{u_x \leq W(t)} \ell(\mathbf{u} - \mathbf{i}W) f_0(\mathbf{u}) d\mathbf{u} - \int_{u_x \geq W(t)} \ell(\mathbf{u} - \mathbf{i}W) f_0(\mathbf{u}) d\mathbf{u} \right), \end{aligned} \quad (3.1)$$

where $C = |\partial\Omega_L|$. By [7, Lemma 2.8], $F_0(W)$ is a positive, increasing C^1 function of W . Let $R_W(t) = F(t) - F_0(W(t))$ be the force due to the collisions with the particles that occurred before time t (“precollisions”) if the body were to move with velocity $W(t)$, given by the formula

$$\begin{aligned} R_W(t) &= \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{u_x \geq W(t)} d\mathbf{u} \ell(\mathbf{u} - \mathbf{i}W(t)) \{f_0(\mathbf{u}) - f_-(t, \mathbf{x}, \mathbf{u})\} \\ &\quad + \int_{\partial\Omega_R(t)} dS_{\mathbf{x}} \int_{u_x \leq W(t)} d\mathbf{u} \ell(\mathbf{u} - \mathbf{i}W(t)) \{f_-(t, \mathbf{x}, \mathbf{u}) - f_0(\mathbf{u})\}. \end{aligned}$$

As mentioned in the introduction, we then define a new motion V_W by means of the equation

$$\frac{dV_W}{dt} = \frac{F_0(V_\infty) - F_0(W(t))}{V_\infty - W(t)} (V_\infty - V_W) - R_W(t). \quad (3.2)$$

The main part of the proof is to establish, as stated in Theorem 3.1 below, an upper and a lower bound of $R_W(t)$ for all $W \in \mathcal{W}$. Using Theorem 3.1, we will prove by means of Lemma 3.1 that $V_W \in \mathcal{W}$.

Definition 1 (Class of possible motions for the irreversible case). *We define \mathcal{W} as the family of functions W that satisfy the following conditions.*

- (i) $W : [0, \infty) \rightarrow \mathbb{R}$ is Lipschitz and $W(0) = V_\infty + \gamma$.
- (ii) W is decreasing over the interval $[0, t_0]$ for $t_0 = |\ln \gamma|$.
- (iii) For all $W \in \mathcal{W}$, $t \in [0, \infty)$ and $\gamma \in (0, 1)$,

$$\gamma h(t, \gamma) \leq V_\infty - W(t) \leq \gamma g(t, \gamma), \quad (3.3)$$

that is,

$$V_\infty - \gamma g(t, \gamma) \leq W(t) \leq V_\infty - \gamma h(t, \gamma),$$

where

$$\begin{aligned} -g(t, \gamma) &= e^{-B_0 t} + \frac{\gamma^p A_+}{t^{p+d}} \chi \{t \geq t_0 + 1\}, \\ -h(t, \gamma) &= e^{-B_\infty t} + \frac{\gamma^p A_-}{\langle t \rangle^{p+d}}. \end{aligned}$$

with

$$B_0 = \max_{V \in [V_\infty - \gamma, V_\infty + \gamma]} F'_0(V), B_\infty = \min_{V \in [V_\infty - \gamma, V_\infty + \gamma]} F'_0(V).$$

Theorem 3.1. *If k and a_0 satisfy the Irreversible Criterion, then for all small enough γ , there exists c_1 , C_2 , and $C > 0$ such that for all $W \in \mathcal{W}$, we have*

$$R_W(t) \leq \left[-c_1 \frac{\gamma^{p+1}}{t^{p+d}} + \frac{C_2 \gamma^{p+1} A_-^{p+1}}{\langle t \rangle^{(p+d)(p+1)}} \right] \chi \{t \geq t_0\} \leq 0 \quad (3.4)$$

and

$$R_W(t) \geq -\frac{C (\gamma + \gamma^{p+1} A_-)^{p+1}}{\langle t \rangle^{p+d}}. \quad (3.5)$$

Proof. We postpone the proof of Theorem 3.1 to the end of Section 3. \square

Lemma 3.1. *If k and a_0 satisfy the Irreversible Criterion, for small enough γ , we can choose A_+ and A_- in Definition 1 such that, for any $W \in \mathcal{W}$, the solution V_W to the iteration equation (3.2)*

$$\frac{dV_W}{dt} = Q(t) (V_\infty - V_W) - R_W(t), \quad Q(t) = \frac{F_0(V_\infty) - F_0(W(t))}{V_\infty - W(t)},$$

satisfies

$$-\gamma e^{-B_\infty t} - \frac{A_- \gamma^{p+1}}{(1+t)^{p+d}} \leq V_\infty - V_W(t) \leq -\gamma e^{-B_0 t} - \frac{A_+ \gamma^{p+1}}{t^{p+d}} \chi \{t \geq t_0 + 1\} < 0.$$

In other words, for every $W \in \mathcal{W}$, we have $V_W \in \mathcal{W}$.

Proof. By (3.2) we have

$$\frac{d(V_\infty - V_W)}{dt} = -Q(t) (V_\infty - V_W) + R_W(t),$$

hence

$$V_\infty - V_W(t) = -\gamma e^{-\int_0^t Q(r) dr} + \int_0^t \left(e^{-\int_s^t Q(r) dr} \right) R_W(s) ds$$

because $V_\infty - V_W(0) = -\gamma$. On the one hand, by (3.5),

$$\begin{aligned} V_\infty - V_W(t) &\geq -\gamma e^{-B_\infty t} - \int_0^t \left(e^{-\int_s^t Q(r) dr} \right) \frac{C (\gamma + \gamma^{p+1} A_-)^{p+1}}{\langle s \rangle^{p+d}} ds \\ &\geq -\gamma e^{-B_\infty t} - \int_0^t e^{-B_\infty(t-s)} \frac{C (\gamma + \gamma^{p+1} A_-)^{p+1}}{\langle s \rangle^{p+d}} ds \\ &= -\gamma e^{-B_\infty t} - C (\gamma + \gamma^{p+1} A_-)^{p+1} \int_0^t \frac{e^{-B_\infty(t-s)}}{\langle s \rangle^{p+d}} ds \end{aligned}$$

where

$$\begin{aligned}
& \int_0^t \frac{e^{-B_\infty(t-s)}}{\langle s \rangle^{p+d}} ds \\
&= \int_0^{\frac{t}{2}} e^{-B_\infty(t-s)} \frac{1}{(1+s)^{p+d}} ds + \int_{\frac{t}{2}}^t e^{-B_\infty(t-s)} \frac{1}{(1+s)^{p+d}} ds \\
&\leq \int_0^{\frac{t}{2}} e^{-B_\infty(t-s)} ds + \frac{C}{(1+t)^{p+d}} \int_{\frac{t}{2}}^t e^{-B_\infty(t-s)} ds \\
&\leq \frac{1}{B_\infty} (e^{-\frac{B_\infty t}{2}} - e^{-B_\infty t}) + \frac{C}{(1+t)^{p+d}} \leq \frac{C'}{(1+t)^{p+d}},
\end{aligned}$$

That is,

$$V_\infty - V_W(t) \geq -\gamma e^{-B_\infty t} - \frac{C'(\gamma + \gamma^{p+1} A_-)^{p+1}}{(1+t)^{p+d}}.$$

Letting $A_- > C'$, we have $C'(1 + \gamma^p A_-)^{p+1} \leq A_-$ for small γ , whence

$$V_\infty - V_W(t) \geq -\gamma e^{-B_\infty t} - \frac{A_- \gamma^{p+1}}{(1+t)^{p+d}}.$$

On the other hand, with the aforementioned A_- , by (3.4), we have for small γ

$$R_W(t) \leq -c \frac{\gamma^{p+1}}{t^{p+d}} \chi\{t \geq t_0\} \leq 0.$$

Thus

$$\begin{aligned}
V_\infty - V_W(t) &\leq -\gamma e^{-B_0 t} - c \gamma^{p+1} \int_0^t \left(e^{-\int_s^t Q(r) dr} \right) \frac{1}{s^{p+d}} \chi\{s \geq t_0\} ds \\
&\leq -\gamma e^{-B_0 t} - c \gamma^{p+1} \int_{t_0}^t e^{-B_0(t-s)} \frac{1}{s^{p+d}} ds,
\end{aligned}$$

as long as $1 + t_0 < t$, where

$$\int_{t_0}^t e^{-B_0(t-s)} \frac{1}{s^{p+d}} ds \geq \int_{t-1}^t e^{-B_0(t-s)} \frac{1}{s^{p+d}} ds \geq e^{-B_0} \frac{1}{t^{p+d}}.$$

Hence

$$V_\infty - V_W(t) \leq -\gamma e^{-B_0 t} - \frac{c \gamma^{p+1}}{t^{p+d}} \chi\{t \geq t_0 + 1\}.$$

Therefore, selecting $A_+ \leq c$ yields

$$V_\infty - V_W(t) \leq -\gamma e^{-B_0 t} - \frac{A_+ \gamma^{p+1}}{t^{p+d}} \chi\{t \geq t_0 + 1\}.$$

□

Proof of Theorem 1.1 (a). The proof is almost identical to that in [3], so we merely sketch it here. Let $L = \max\{V_\infty + 1, E + F_0(V_\infty + 1) + C\gamma^{p+1}\}$ and $\mathcal{K} = \{W \in \mathcal{W} \mid \sup(|W(t)| + |W'(t)|) \leq L\}$. Then \mathcal{K} is a compact convex set in $C([0, \infty))$. We define an operator $\mathcal{A} : W \rightarrow V_W$, where V_W is defined in (3.2). Then \mathcal{A} maps \mathcal{K} into itself by Lemma 3.1. By the Schauder fixed point theorem it suffices to prove that \mathcal{A} is continuous in the topology of $C([0, \infty))$.

In order to accomplish that task, we let $W_j \rightarrow W$ in $C([0, \infty))$, where $W_j \in \mathcal{K}$. Fix $T > 0$ so large that the interval (T, ∞) provides a negligible contribution due to the uniform decay in Theorem 3.1. Let N be a positive integer. Define A_j^N be the set of all pairs (x, v_x) such that no trajectory passing through (T, x, v_x) has collided more than N times in the time interval $[0, T]$. Let B_j^N be its complement. We write $R_{W_j}(t) = R_{W_j}(t; A_j^N) + R_{W_j}(t; B_j^N)$. By Theorem 3.1 we have $|R_{W_j}(t; B_j^N)| \leq (C\gamma^{p+1})^N$ for $t \leq T$, while $R_{W_j}(t; A_j^N) \rightarrow R_W(t; A^N)$ in $C([0, T])$ by N uses of the boundary condition. It follows that $R_{W_j}(t) \rightarrow R_W(t)$ uniformly in $[0, T]$. □

Proof of Theorem 1.1 (b). If (V, f) is a solution in the sense of Theorem 1.1, then it is a fixed point of \mathcal{A} , so that Theorem 3.1 is valid for it. We need only check that the strict inequalities

$$\gamma e^{-B_0 t} + \frac{A_+ \gamma^{p+1}}{t^{p+d}} \chi \{t \geq t_0 + 1\} < V(t) - V_\infty < \gamma e^{-B_\infty t} + \frac{A_- \gamma^{p+1}}{(1+t)^{p+d}}, \quad (3.6)$$

are valid for small $t > 0$. Indeed, note that at $t = 0$ we have $\gamma = V(0) - V_\infty < \gamma + C\gamma^{p+1}$ and $V'(0) = E - F(0) = F_0(V_\infty) - F_0(V_\infty + \gamma) < -\gamma \min(F'_0) \leq -\gamma B_\infty < 0$. Therefore (3.6) is valid for very small $t > 0$ and hence for all $t > 0$. \square

3.2. Properties of \mathcal{W} . In this subsection we begin the proof of Theorem 3.1. We split $R_W(t) = r_W^L(t) + r_W^R(t)$, where the left contribution $r_W^L(t)$ and the right contribution $r_W^R(t)$ are given by

$$\begin{aligned} r_W^L(t) &= \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{u_x \geq W(t)} d\mathbf{u} \ell(\mathbf{u} - \mathbf{i}W(t)) \{f_0(\mathbf{u}) - f_-(t, \mathbf{x}, \mathbf{u})\}, \\ r_W^R(t) &= \int_{\partial\Omega_R(t)} dS_{\mathbf{x}} \int_{u_x \leq W(t)} d\mathbf{u} \ell(\mathbf{u} - \mathbf{i}W(t)) \{f_-(t, \mathbf{x}, \mathbf{u}) - f_0(\mathbf{u})\}. \end{aligned}$$

We shall first prove some properties of the iteration family \mathcal{W} defined in Definition 1. Then we shall estimate $r_W^L(t)$ and $r_W^R(t)$ in Lemmas 3.3 and 3.4, from which Theorem 3.1 will follow.

For any function $Y : [0, \infty) \rightarrow \mathbb{R}$, we denote its average over time intervals by

$$\langle Y \rangle_{s,t} = \frac{1}{t-s} \int_s^t Y(\tau) d\tau, \quad \langle Y \rangle_{0,t} = \langle Y \rangle_t.$$

Thus $Y \in L^1(\mathbb{R})$ implies $\langle Y \rangle_t = O(1/t)$ for large t . The family $\mathcal{W} = \{W\}$, defined in Definition 1 for the irreversible case, has the following properties.

Lemma 3.2. *Let \mathcal{W} be defined in Definition 1 for the irreversible case. For all small enough γ and hence for all large enough t_0 , we have*

$$\langle W \rangle_t - W(t) \geq \frac{C\gamma}{t} \text{ for } t \geq t_0,$$

and

$$\langle W \rangle_t - W(t) \leq \frac{C(\gamma + \gamma^{p+1} A_-)}{1+t}, \text{ for all } t \geq 0.$$

Proof. On the one hand,

$$\begin{aligned} \langle W \rangle_t - W(t) &= \frac{1}{t} \int_0^t W(s) ds - W(t) \\ &\geq \frac{1}{t} \int_0^t (V_\infty + \gamma e^{-B_0 s}) ds - \left(V_\infty + \gamma e^{-B_\infty t} + \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}} \right) \\ &\geq \frac{C_1 \gamma}{t} - \gamma e^{-B_\infty t} - \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}} \end{aligned}$$

Notice that, for all small enough γ and hence all large enough t_0 , the second and the third terms are absorbed into the first term for $t \geq t_0$. So

$$\langle W \rangle_t - W(t) \geq \frac{C\gamma}{t} \chi \{t \geq t_0\}.$$

On the other hand,

$$\begin{aligned} \langle W \rangle_t - W(t) &= \frac{1}{t} \int_0^t W(s) ds - W(t) \\ &\leq \frac{1}{t} \int_0^t \left(V_\infty + \gamma e^{-B_\infty s} + \frac{\gamma^{p+1} A_-}{\langle s \rangle^{p+d}} \right) ds - \left(V_\infty + \gamma e^{-B_0 t} + \frac{\gamma^{p+1} A_+}{t^{p+d}} \chi \{t \geq t_0 + 1\} \right) \\ &\leq \frac{1}{t} \int_0^t \left(\gamma e^{-B_\infty s} + \frac{\gamma^{p+1} A_-}{\langle s \rangle^{p+d}} \right) ds \leq \frac{C(\gamma + \gamma^{p+1} A_-)}{1+t}. \end{aligned}$$

\square

Corollary 3.1. *For small enough γ , we have*

- (i) $\langle W \rangle_t > W(t)$ for all t .
- (ii) $\langle W \rangle_t$ is a decreasing function.
- (iii) $\langle W \rangle_t > \langle W \rangle_{s,t}$, $\forall s \in (0, t)$.

Proof. For $t \leq t_0$, (i) follows from the assumption that W is decreasing over the interval $[0, t_0]$. For $t \geq t_0$, we quote Lemma 3.2.

(ii)

$$\frac{d}{dt} \langle W \rangle_t = \frac{1}{t} (-\langle W \rangle_t + W(t)) < 0.$$

(iii)

$$\begin{aligned} \langle W \rangle_{s,t} - \langle W \rangle_t &= \frac{1}{t-s} \int_s^t W(\tau) d\tau - \frac{1}{t} \int_0^t W(\tau) d\tau \\ &= \frac{1}{t-s} \int_0^t W(\tau) d\tau - \frac{1}{t-s} \int_0^s W(\tau) d\tau - \frac{1}{t} \int_0^t W(\tau) d\tau \\ &= \frac{s}{t-s} \left(\frac{1}{t} \int_0^t W(\tau) d\tau - \frac{1}{s} \int_0^s W(\tau) d\tau \right) < 0, \text{ by (ii).} \end{aligned}$$

□

Since $\langle W \rangle_{s,t}$ is a continuous function of s and t , the existence of a precollision at some time earlier than t requires that the velocity satisfies

$$v_x \in \left[\inf_{s < t} \langle W \rangle_{s,t}, \sup_{s < t} \langle W \rangle_{s,t} \right] = \left[\inf_{s < t} \langle W \rangle_{s,t}, \langle W \rangle_t \right], \quad (3.7)$$

by (iii) of Corollary 3.1. We estimate $\inf_{s < t} \langle W \rangle_{s,t}$ by

$$\langle W \rangle_{s,t} \geq V_\infty - \gamma \langle g \rangle_{s,t} \geq V_\infty \quad (3.8)$$

since $g \leq 0$. It follows that

$$W(t) - \inf_{s < t} \langle W \rangle_{s,t} \leq V_\infty - \gamma h(t, \gamma) - V_\infty = -\gamma h(t, \gamma) = \gamma e^{-B_\infty t} + \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}}. \quad (3.9)$$

3.3. The Left Side. In the next lemma we estimate the force on the left side of the cylinder.

Lemma 3.3. *Let $W \in \mathcal{W}$ be defined in Definition 1 and let K and a_0 satisfy the Assumptions A1-A4. If k and a_0 satisfy the Irreversal Criterion, then for all sufficiently small γ we have the inequalities*

$$-\frac{C(\gamma + \gamma^{p+1} A_-)^{p+1}}{(1+t)^{p+d}} \leq r_W^L(t) \leq -c \frac{\gamma^{p+1}}{t^{d+p}} \chi\{t \geq t_0\}.$$

Proof. To establish upper and lower bounds of $-r_W^L(t)$, we need upper and lower bounds of $f_+(t, \mathbf{x}; \mathbf{v})$. Recall the boundary condition on the left end of the cylinder

$$f_+(t, \mathbf{x}; \mathbf{v}) = \int_{u_x \geq W(t)} K(\mathbf{v} - \mathbf{i}W(t); \mathbf{u} - \mathbf{i}W(t)) f_-(t, \mathbf{x}; \mathbf{u}) d\mathbf{u}.$$

Motivated by condition (3.7), we write the precollision characteristic function as

$$\begin{aligned} \chi_0(t, \mathbf{u}) &= \chi \left\{ \mathbf{u} : \forall s \in (0, t), \text{ either } u_x \neq \langle W \rangle_{s,t} \text{ or } |u_\perp| > \frac{2r}{t-s} \right\}, \\ \chi_1(t, \mathbf{u}) &= \chi \left\{ \mathbf{u} : \exists s \in (0, t) \text{ s.t. } u_x = \langle W \rangle_{s,t} \text{ and } |u_\perp| \leq \frac{2r}{t-s} \right\}. \end{aligned}$$

We observe that if the precollisions occurred at a sequence of earlier times t_j converging to t , it would then follow that $v_x = W(t)$, so there would be no contribution to the force since $\ell(0) = 0$. In light of this observation, we can always assume that there is a first precollision, that is, a collision that occurs at an earlier time closest to t . In such a case, let τ be the time and $\boldsymbol{\xi}$ be the position of that first precollision. Of course, τ and $\boldsymbol{\xi}$ depend on $t, \mathbf{x}, \mathbf{u}$. These notations enable us to write

$$f_-(t, \mathbf{x}; \mathbf{u}) = f_+(\tau, \boldsymbol{\xi}; \mathbf{u}) \chi_1(t, \mathbf{u}) + f_0(\mathbf{u}) \chi_0(t, \mathbf{u}). \quad (3.10)$$

Putting (3.10) into the boundary condition, we have

$$\begin{aligned} f_+(t, \mathbf{x}; \mathbf{v}) &= \int_{u_x \geq W(t)} K(\mathbf{v} - \mathbf{i}W(t); \mathbf{u} - \mathbf{i}W(t)) \\ &\quad \times [f_+(\tau, \boldsymbol{\xi}; \mathbf{u})\chi_1(t, \mathbf{u}) + f_0(\mathbf{u})\chi_0(t, \mathbf{u})]d\mathbf{u}. \end{aligned}$$

Because the momentum is only transferred horizontally, we may rewrite this equation as

$$\begin{aligned} a_+(t, \mathbf{x}; v_x)b(v_\perp) &= b(v_\perp) \int_{u_x \geq W(t)} k(v_x - W(t), u_x - W(t)) \\ &\quad \times \{a_+(\tau, \boldsymbol{\xi}; u_x)b(u_\perp)\chi_1(t, \mathbf{u}) + a_0(u_x)b(u_\perp)\chi_0(t, \mathbf{u})\}d\mathbf{u}. \end{aligned} \quad (3.11)$$

Since $b(v_\perp)$ could possibly vanish, we do not divide by $b(v_\perp)$ on both sides. In order to get a lower bound of $f_+(t, \mathbf{x}; \mathbf{v})$, we notice that

$$\begin{aligned} f_+(t, \mathbf{x}; \mathbf{v}) &= a_+(t, \mathbf{x}; v_x)b(v_\perp) \\ &\geq b(v_\perp) \int_{u_x \geq W(t)} k(v_x - W(t), u_x - W(t))a_0(u_x)b(u_\perp)\chi_0(t, \mathbf{u})d\mathbf{u} \\ &\geq b(v_\perp) \int_{V_\infty + 2\gamma}^{+\infty} k(v_x - W(t), u_x - W(t))a_0(u_x)du_x, \end{aligned}$$

where we used in the last line the inequality

$$W(t) \leq V_\infty + \gamma e^{-B_\infty t} + \frac{\gamma^{p+1}A_-}{\langle t \rangle^{p+d}} \leq V_\infty + 2\gamma.$$

Thus for some s we have

$$u_x = \langle W \rangle_{s,t} \leq V_\infty + 2\gamma$$

so that $\chi_0(t, \mathbf{u}) = 1$. Now recall the Irreversal Criterion

$$\int_{u_x \geq V_\infty} k(0, u_x - V_\infty)a_0(u_x)du_x > a_0(V_\infty).$$

It follows from A.4 that there exists $\delta > 0$ such that

$$\inf_{\substack{t, x \in \partial\Omega(t) \\ v_x \in [V_\infty - 2\gamma, V_\infty + 2\gamma]}} \int_{u_x \geq V_\infty + 2\gamma} k(v_x - W(t), u_x - W(t))a_0(u_x)du_x \geq a_0(V_\infty) + \delta,$$

for all small enough γ . Hence we have obtained the lower bound

$$f_+(t, \mathbf{x}; \mathbf{v}) \geq (f_0(\mathbf{v}) + \delta b(v_\perp)) \quad \text{for } |v_x - V_\infty| \leq 2\gamma. \quad (3.12)$$

To gain an upper bound of $f_+(t, \mathbf{x}; \mathbf{v})$, we return to (3.11) and observe that

$$\begin{aligned} a_+(t, \mathbf{x}; v_x)b(v_\perp) &\leq a_+^* \int_{W(t)}^{\langle W \rangle_t} k(v_x - W(t), u_x - W(t))du_x \\ &\quad + b(v_\perp) \int_{u_x \geq W(t)} k(v_x - W(t), u_x - W(t))a_0(u_x)du_x \end{aligned}$$

by (3.7), where

$$a_+^* = \sup \{a_+(\tau, \xi; u_x) \mid \xi \in \partial\Omega(\tau), \tau \in [0, \infty), \text{ and } u_x \in [V_\infty - 2\gamma, V_\infty + 2\gamma]\}. \quad (3.13)$$

By the fact that $\langle W \rangle_t - W(t) \leq C\gamma$, proven in Lemma 3.2, we have

$$a_+(t, \mathbf{x}; v_x)b(v_\perp) \leq b(v_\perp)C\gamma a_+^* + b(v_\perp)C,$$

using Assumptions A2 and A4. Thus taking the supremum over all times $t \in [0, \infty)$, positions $x \in \partial\Omega(t)$ and velocities $v_x \in [V_\infty - 2\gamma, V_\infty + 2\gamma]$, we have

$$b(v_\perp)a_+^* \leq b(v_\perp)C\gamma a_+^* + Cb(v_\perp).$$

That is,

$$b(v_\perp)a_+^* \leq \frac{Cb(v_\perp)}{1 - C\gamma} \leq Cb(v_\perp), \quad (3.14)$$

for $\gamma < \frac{1}{C}$. This is an upper bound for $f_+(t, \mathbf{x}; \mathbf{v})$.

We are now ready to establish upper and lower bounds of $-r_W^L(t)$ for the irreversible case. We begin with the crucial lower bound of $-r_W^L(t)$ because that is the main reason why $r_W^L + r_W^R \leq 0$. Using the lower bound (3.12) of $f_+(\tau, \xi, \mathbf{u})$, we get

$$\begin{aligned}
-r_W^L(t) &= \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{u_x \geq W(t)} d\mathbf{u} \ell(u_x - W(t)) \{f_-(t, \mathbf{x}, \mathbf{u}) - f_0(\mathbf{u})\} \\
&= \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{u_x \geq W(t)} d\mathbf{u} \ell(u_x - W(t)) [f_+(\tau, \xi, \mathbf{u}) \chi_1(t, \mathbf{u}) + f_0(\mathbf{u}) \chi_0(t, \mathbf{u}) - f_0(\mathbf{u})] \\
&= \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} \int_{W(t)}^{\langle W \rangle_t} du_x \ell(u_x - W(t)) (f_+(\tau, \xi, \mathbf{u}) - f_0(\mathbf{u})) \\
&\geq \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} b(u_{\perp}) \int_{W(t)}^{\langle W \rangle_t} du \ell(u_x - W(t)) (a_0(V_{\infty}) + \delta - a_0(u_x))
\end{aligned} \tag{3.15}$$

because $u_x = \langle W \rangle_{\tau, t} \leq \langle W \rangle_t$ for some τ . For small enough γ , by continuity of a_0 we have

$$a_0(V_{\infty}) + \delta - a_0(u_x) \geq \frac{\delta}{2} > 0.$$

We then infer via A3 that

$$-r_W^L(t) \geq C \frac{\delta}{2} \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} b(u_{\perp}) du_{\perp} \int_{W(t)}^{\langle W \rangle_t} du_x \ell(u_x - W(t)) \geq C \frac{\delta (\langle W \rangle_t - W(t))^{p+1}}{(1+t)^{d-1}} \geq 0.$$

We note that in this expression the integral over u_{\perp} is at least Ct^{1-d} because $b(0) > 0$. Furthermore, as proven in Lemma 3.2, we have

$$\langle W \rangle_t - W(t) \geq \frac{C\gamma}{t} \chi\{t \geq t_0\},$$

whence

$$-r_W^L(t) \geq C \frac{\gamma^{p+1}}{t^{d+p}} \chi\{t \geq t_0\}$$

for small enough γ . This is the desired lower bound of $-r_W^L$.

By the upper bound (3.14) of $f_+(\tau, \xi, \mathbf{u})$ and Lemma 3.2, we now determine an upper bound for $-r_W^L$. Indeed, by (3.15),

$$\begin{aligned}
|-r_W^L(t)| &= \left| \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{W(t)}^{\langle W \rangle_t} du_x \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} \ell(u_x - W(t)) [f_+(\tau, \xi, \mathbf{u}) - f_0(\mathbf{u})] \right| \\
&\leq \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{W(t)}^{\langle W \rangle_t} du_x \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} \ell(u_x - W(t)) [f_+(\tau, \xi, \mathbf{u}) + f_0(\mathbf{u})] \\
&\leq C \int_{W(t)}^{\langle W \rangle_t} du_x \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} \ell(u_x - W(t)) b(u_{\perp}).
\end{aligned}$$

Splitting the integral according to the regions $\tau < t/2$ and $\tau \geq t/2$, we have

$$\begin{aligned}
|r_W^L(t)| &\leq \frac{C (\langle W \rangle_t - W(t))^{p+1}}{(1+t)^{d-1}} + C \int_{W(t)}^{\langle W \rangle_t} du_x \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}, \tau \geq \frac{t}{2}} du_{\perp} \ell(u_x - W(t)) b(u_{\perp}) \\
&= I + II.
\end{aligned}$$

By Assumption A3 and Lemma 3.2,

$$I \leq \frac{C \left(\frac{C(\gamma + \gamma^{p+1} A_-)}{1+t} \right)^{p+1}}{(1+t)^{d-1}} \leq C \frac{(\gamma + \gamma^{p+1} A_-)^{p+1}}{(1+t)^{d+p}}.$$

For the second term II , by the precollision condition (3.7) we notice that

$$\begin{aligned}
u_x &= \langle W \rangle_{\tau, t} \leq V_{\infty} - \gamma \langle h \rangle_{\tau, t} \\
&\leq V_{\infty} + \sup_{\frac{t}{2} \leq \tau \leq t} \frac{\gamma}{t-\tau} \int_{\tau}^t \left(e^{-B_{\infty} r} + \frac{\gamma^p A_-}{\langle r \rangle^{p+d}} \right) dr \\
&\leq V_{\infty} + CM(t),
\end{aligned}$$

where

$$M(t) = \gamma e^{-B_\infty \frac{t}{2}} + \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}} \leq \frac{\gamma + \gamma^{p+1} A_-}{\langle t \rangle^{p+d}}.$$

With Assumption A3, the above inequality allows us to estimate the second term as

$$\begin{aligned} II &\leq C \int_{W(t)}^{V_\infty + CM(t)} du_x \int_{|u_\perp| \leq \frac{2r}{t-\tau}, \tau \geq \frac{t}{2}} du_\perp \ell(u_x - W(t)) b(u_\perp) \\ &\leq C \int_0^{V_\infty + CM(t) - W(t)} |z|^p dz \leq C [V_\infty - W(t) + CM(t)]^{p+1} \leq C [M(t)]^{p+1} \\ &\leq C \left(\frac{\gamma + \gamma^{p+1} A_-}{\langle t \rangle^{p+d}} \right)^{p+1} \end{aligned}$$

because $V_\infty - W(t) \leq 0$. Putting I and II together, we have

$$|r_W^L(t)| \leq C \frac{(\gamma + \gamma^{p+1} A_-)^{p+1}}{(1+t)^{d+p}},$$

which is the claimed upper bound for $-r_W^L(t)$. \square

3.4. The Right Side. We now proceed to bound the force $|r_W^R|$ on the right side of the cylinder.

Lemma 3.4. *Under the same assumptions as in Lemma 3.3, we have*

$$|r_W^R(t)| \leq \frac{C \gamma^{p+1} A_-^{p+1}}{t^{(p+d)(p+1)}} \chi\{t \geq t_0\}.$$

Proof. We first notice that $r_W^R(t) = 0$ for all $t \leq t_0$ because W is decreasing. In fact, suppose that on the right there is a precollision at time τ and a later collision at time $t \leq t_0$. If the velocity of the particle in the time period (τ, t) is u_x , then $u_x \geq W(\tau)$ and $u_x \leq W(t) < W(\tau)$ which is a contradiction.

Taking $t \geq t_0$ and recalling the boundary condition on the right side of the cylinder, we have

$$f_+(t, \mathbf{x}; \mathbf{v}) = \int_{u_x \leq W(t)} K(\mathbf{v} - \mathbf{i}W(t); \mathbf{u} - \mathbf{i}W(t)) f_-(t, \mathbf{x}; \mathbf{u}) d\mathbf{u}.$$

Plugging in the precollision condition (3.10) again, namely

$$f_-(t, \mathbf{x}; \mathbf{u}) = f_+(\tau, \boldsymbol{\xi}; \mathbf{u}) \chi_1(t, \mathbf{u}) + f_0(\mathbf{u}) \chi_0(t, \mathbf{u}),$$

we have

$$\begin{aligned} f_+(t, \mathbf{x}; \mathbf{v}) &= \int_{u_x \leq W(t)} K(\mathbf{v} - \mathbf{i}W(t); \mathbf{u} - \mathbf{i}W(t)) \\ &\quad \times \{f_+(\tau, \boldsymbol{\xi}; \mathbf{u}) \chi_1(t, \mathbf{u}) + f_0(\mathbf{u}) \chi_0(t, \mathbf{u})\} d\mathbf{u} \\ &= b(v_\perp) \int_{u_x \leq W(t)} k(v_x - W(t), u_x - W(t)) \\ &\quad \times \{a_+(\tau, \boldsymbol{\xi}; u_x) b(u_\perp) \chi_1(t, \mathbf{u}) + a_0(u_x) b(u_\perp) \chi_0(t, \mathbf{u})\} d\mathbf{u}. \end{aligned}$$

We then estimate

$$\begin{aligned} f_+(t, \mathbf{x}; \mathbf{v}) &\leq b(v_\perp) \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} k(v_x - W(t), u_x - W(t)) a_+(\tau, \boldsymbol{\xi}; u_x) du_x \\ &\quad + b(v_\perp) \int_{-\infty}^{W(t)} k(v_x - W(t), u_x - W(t)) a_0(u_x) du_x \end{aligned}$$

Assumption A4 takes care of the second term. To estimate the first term, we must control the size of $W(t) - \inf_{s < t} \langle W \rangle_{s,t}$. Recalling (3.9),

$$W(t) - \inf_{s < t} \langle W \rangle_{s,t} \leq \gamma e^{-B_\infty t} + \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}},$$

we estimate the first term by

$$b(v_\perp) \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} k(v_x - W(t), u_x - W(t)) a_+(\tau, \boldsymbol{\xi}; u_x) du_x \leq b(v_\perp) C \gamma a_+^*.$$

where a_+^* is defined in (3.13). Thus, taking supremums as in the earlier estimate (3.14), we have $b(v_\perp)a_+^* \leq b(v_\perp)C\gamma a_+^* + Cb(v_\perp)$. Since γ is small, we deduce that

$$b(v_\perp)a_+^* \leq Cb(v_\perp).$$

With this upper bound of $f_+(\tau, \xi, u)$, we arrive at

$$\begin{aligned} |r_W^R(t)| &= \left| \int_{\partial\Omega_R(t)} dS_x \int_{u_x \leq W(t)} d\mathbf{u} \ell(u_x - W(t)) [f_-(t, \mathbf{x}; \mathbf{u}) - f_0(\mathbf{u})] \right| \\ &= \left| \int_{\partial\Omega_R(t)} dS_x \int_{u_x \leq W(t)} d\mathbf{u} \ell(u_x - W(t)) [f_+(\tau, \xi; \mathbf{u})\chi_1(t, \mathbf{u}) + f_0(\mathbf{u})\chi_0(t, \mathbf{u}) - f_0(\mathbf{u})] \right| \\ &= \left| \int_{\partial\Omega_R(t)} dS_x \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} d\mathbf{u} \ell(u_x - W(t)) [f_+(\tau, \xi; \mathbf{u}) - f_0(\mathbf{u})] \right| \\ &\leq \int_{\partial\Omega_R(t)} dS_x \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} d\mathbf{u} \ell(u_x - W(t)) [f_+(\tau, \xi; \mathbf{u}) + f_0(\mathbf{u})] \\ &\leq C \int_{|u_\perp| \leq \frac{2r}{t-\tau}} du_\perp \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} du_x \ell(u_x - W(t)) b(u_\perp). \end{aligned}$$

As before, we split the integral at $\tau = t/2$. We deal with the $\tau < \frac{t}{2}$ part first (although it is not the main contribution unless $d = 1$). We have

$$\begin{aligned} I &= C \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} du_x \int_{|u_\perp| \leq \frac{2r}{t-\tau}, \tau < \frac{t}{2}} du_\perp \ell(u_x - W(t)) b(u_\perp) \leq C \frac{(W(t) - \inf_{s < t} \langle W \rangle_{s,t})^{p+1}}{(1+t)^{d-1}} \\ &\leq \frac{C}{(1+t)^{d-1}} \left(\gamma e^{-B_\infty t} + \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}} \right)^{p+1} \end{aligned}$$

for $t \geq t_0$ by (3.9).

For the $\tau \in [\frac{t}{2}, t]$ part, which is the major contribution, we know

$$\begin{aligned} u_x &= \langle W \rangle_{\tau,t} \geq V_\infty - \gamma \langle g \rangle_{\tau,t} \\ &\geq V_\infty + \inf_{\frac{t}{2} \leq \tau \leq t} \frac{\gamma}{t-\tau} \int_\tau^t \left(e^{-B_0 r} + \frac{\gamma^p A_+}{r^{p+d}} \chi\{r \geq t_0 + 1\} \right) dr \\ &\geq V_\infty \end{aligned}$$

which yields, for $t \geq t_0$,

$$\begin{aligned} II &\leq C \int_{V_\infty}^{W(t)} du_x \int_{|u_\perp| \leq \frac{2r}{t-\tau}, \frac{t}{2} \leq \tau \leq t} du_\perp \ell(u_x - W(t)) b(u_\perp) \\ &\leq C \int_{V_\infty - W(t)}^0 |u_x - W(t)|^p du_x \leq C |V_\infty - W(t)|^{p+1} \\ &\leq C \left| \gamma e^{-B_\infty t} + \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}} \right|^{p+1}, \end{aligned}$$

by (3.3). Collecting the estimates for I and II , we have

$$\begin{aligned} |r_W^R(t)| &\leq \frac{C}{(1+t)^{d-1}} \left(\gamma e^{-B_\infty t} + \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}} \right)^{p+1} \chi\{t \geq t_0\} \\ &\quad + C \left(\gamma e^{-B_\infty t} + \frac{\gamma^{p+1} A_-}{\langle t \rangle^{p+d}} \right)^{p+1} \chi\{t \geq t_0\} \\ &\leq \frac{C \gamma^{p+1} A_-^{p+1}}{\langle t \rangle^{(p+d)(p+1)}} \chi\{t \geq t_0\} \end{aligned}$$

since $d \geq 1$. This concludes the proof of Lemma 3.4. \square

Finally, collecting Lemmas 3.3 and 3.4, we have both

$$R_W(t) \leq \left[-c \frac{\gamma^{p+1}}{t^{p+d}} + C_2 \frac{\gamma^{p+1} A_-^{p+1}}{\langle t \rangle^{(p+d)(p+1)}} \right] \chi \{t \geq t_0\}$$

and

$$\begin{aligned} R_W(t) &\geq -\frac{C_1 (\gamma + \gamma^{p+1} A_-)^{p+1}}{\langle t \rangle^{p+d}} - C_2 \frac{\gamma^{p+1} A_-^{p+1}}{\langle t \rangle^{(p+d)(p+1)}} \chi \{t \geq t_0\} \\ &\geq -\frac{C (\gamma + \gamma^{p+1} A_-)^{p+1}}{\langle t \rangle^{p+d}} \end{aligned}$$

Because γ is small, Theorem 3.1 follows.

4. PROOF OF THE REVERSAL CASE

4.1. Proof assuming the Key Estimate. For the proof of the reversal case, we follow the structure of the proof of the irreversal case in Section 3. Alert reader should keep in mind that we use a class \mathcal{W} different from Definition 1 here for the reversal case. To be specific, the definitions of t_0 , g , and h are different.

Definition 2 (Class of possible motions for the reversal case). *We define \mathcal{W} as the family of functions W which satisfy the following conditions.*

- (i) $W : [0, \infty) \rightarrow \mathbb{R}$ is Lipschitz and $W(0) = V_\infty + \gamma$.
- (ii) W is decreasing over the interval $[0, t_0]$ for $t_0 = K_0 |\ln \gamma|$ with $\frac{1}{B_0} \leq K_0 \leq \frac{2}{B_0}$, where

$$B_0 = \max_{V \in [V_\infty - \gamma, V_\infty + \gamma]} F'_0(V), B_\infty = \min_{V \in [V_\infty - \gamma, V_\infty + \gamma]} F'_0(V).$$

- (iii) For all $W \in \mathcal{W}$, $t \in [0, \infty)$ and $\gamma \in (0, 1)$,

$$\gamma h(t, \gamma) \leq V_\infty - W(t) \leq \gamma g(t, \gamma), \quad (4.1)$$

that is,

$$V_\infty - \gamma g(t, \gamma) \leq W(t) \leq V_\infty - \gamma h(t, \gamma),$$

where

$$\begin{aligned} -g(t, \gamma) &= e^{-B_0 t} - \frac{\gamma^p A_+}{\langle t \rangle^{p+d}}, \\ -h(t, \gamma) &= e^{-B_\infty t} - \frac{\gamma^p A_-}{t^{p+d}} \chi \{t \geq t_0 + 1\}. \end{aligned}$$

Theorem 4.1. *Assume that k and a_0 verify the Reversal Criterion, then for small enough γ , there exists c_1 , C_2 , and $C > 0$ such that for all $W \in \mathcal{W}$, we have*

$$R_W(t) \geq \left(\frac{c_1 \gamma^{p+1}}{t^{p+d}} - \frac{C_2 \gamma^{(1+\varepsilon)(p+1)} A_+^{p+1}}{t^{p+d}} \right) \chi \{t \geq t_0\} \geq 0, \quad (4.2)$$

and

$$R_W(t) \leq \frac{C (\gamma + \gamma^{p+1} A_+)^{p+1}}{(1+t)^{p+d}}. \quad (4.3)$$

Proof. We postpone the proof of Theorem 4.1 to the end of Section 4. \square

Lemma 4.1. *If k and a_0 satisfy the Reversal Criterion, then for small enough γ , we can choose A_+ and A_- in Definition 2 such that, for any $W \in \mathcal{W}$, the solution V_W to the iteration equation (3.2)*

$$\frac{dV_W}{dt} = Q(t) (V_\infty - V_W) - R_W(t), \quad Q(t) = \frac{F_0(V_\infty) - F_0(W(t))}{V_\infty - W(t)},$$

satisfies

$$-\gamma e^{-B_\infty t} + \frac{A_- \gamma^{p+1} \chi \{t \geq t_0 + 1\}}{t^{p+d}} \leq V_\infty - V_W(t) \leq -\gamma e^{-B_0 t} + \frac{A_+ \gamma^{p+1}}{(1+t)^{p+d}}.$$

In other words, for every $W \in \mathcal{W}$, we have $V_W \in \mathcal{W}$.

Proof. By (3.2) we have

$$\frac{d(V_\infty - V_W)}{dt} = -Q(t)(V_\infty - V_W) + R_W(t),$$

hence

$$V_\infty - V_W(t) = -\gamma e^{-\int_0^t Q(r)dr} + \int_0^t \left(e^{-\int_s^t Q(r)dr} \right) R_W(s) ds$$

because $V_\infty - V_W(0) = -\gamma$. On the one hand, by estimate (4.3) we have

$$\begin{aligned} V_\infty - V_W(t) &\leq -\gamma e^{-B_0 t} + \int_0^t e^{-B_\infty(t-s)} \frac{C(\gamma + \gamma^{p+1} A_+)^{p+1}}{(1+s)^{p+d}} ds \\ &= -\gamma e^{-B_0 t} + C(\gamma + \gamma^{p+1} A_+)^{p+1} I, \end{aligned}$$

where

$$\begin{aligned} I &= \int_0^{\frac{t}{2}} e^{-B_\infty(t-s)} \frac{1}{(1+s)^{p+d}} ds + \int_{\frac{t}{2}}^t e^{-B_\infty(t-s)} \frac{1}{(1+s)^{p+d}} ds \\ &\leq \int_0^{\frac{t}{2}} e^{-B_\infty(t-s)} ds + \frac{C}{(1+t)^{p+d}} \int_{\frac{t}{2}}^t e^{-B_\infty(t-s)} ds \\ &\leq \frac{1}{B_\infty} (e^{-\frac{B_\infty t}{2}} - e^{-B_\infty t}) + \frac{C}{(1+t)^{p+d}} \leq \frac{C'}{(1+t)^{p+d}}. \end{aligned}$$

That is,

$$V_\infty - V_W(t) \leq -\gamma e^{-B_0 t} + \frac{C'(\gamma + \gamma^{p+1} A_+)^{p+1}}{(1+t)^{p+d}}.$$

Letting $A_+ > C'$, we have $C'(1 + \gamma^p A_+)^{p+1} \leq A_+$ for small γ , so that

$$V_\infty - V_W(t) \leq -\gamma e^{-B_0 t} + \frac{A_+ \gamma^{p+1}}{(1+t)^{p+d}}.$$

On the other hand, with the aforementioned A_+ , estimate (4.2) reads

$$R_W(t) \geq \frac{c\gamma^{p+1}}{t^{p+d}} \chi\{t \geq t_0\} \geq 0$$

for small γ . Thus

$$\begin{aligned} V_\infty - V_W(t) &\geq -\gamma e^{-B_\infty t} + \int_0^t e^{-B_0(t-s)} \chi\{s \geq t_0\} \frac{c\gamma^{p+1}}{s^{p+d}} ds \\ &= -\gamma e^{-B_\infty t} + c\gamma^{p+1} II, \end{aligned}$$

where

$$II \geq \int_{t-1}^t e^{-B_0} \frac{1}{s^{p+d}} ds \geq e^{-B_0} \frac{1}{t^{p+d}},$$

as long as $1 + t_0 < t$. Hence

$$V_\infty - V_W(t) \geq -\gamma e^{-B_\infty t} + \frac{c^{p+1} \chi\{t \geq t_0 + 1\}}{t^{p+d}}.$$

Therefore, selecting $A_- \leq c'$ yields

$$V_\infty - V_W(t) \geq -\gamma e^{-B_\infty t} + \frac{A_- \gamma^{p+1} \chi\{t \geq t_0 + 1\}}{t^{p+d}}.$$

□

Theorem 1.2 then follows by the same proof as in Theorem 1.1. For the reversal case, we are left with the proof of Theorem 4.1.

4.2. Properties of \mathcal{W} .

Lemma 4.2. *Let \mathcal{W} be defined in Definition 2 for the reversal case. For all small enough γ and hence for all large enough t_0 , we have*

$$\langle W \rangle_t - W(t) \geq \frac{C\gamma}{t} \text{ for } t \geq t_0,$$

and

$$\langle W \rangle_t - W(t) \leq \frac{C(\gamma + \gamma^{p+1}A_+)}{1+t}, \text{ for all } t \geq 0.$$

Proof. On the one hand,

$$\begin{aligned} \langle W \rangle_t - W(t) &= \frac{1}{t} \int_0^t W(s) ds - W(t) \\ &\geq \frac{1}{t} \int_0^t \left(V_\infty + \gamma e^{-B_0 s} - \frac{\gamma^{p+1}A_+}{\langle s \rangle^{p+d}} \right) ds - V_\infty - \gamma e^{-B_\infty t} \\ &\geq \frac{C_1\gamma}{t} - \frac{C_2\gamma^{p+1}}{t} - \gamma e^{-B_\infty t} \end{aligned}$$

For all small enough γ , the second term is absorbed into the first term because $p > 0$. Also, for all small enough γ and hence all large enough t_0 , the third term is absorbed into the first term for $t \geq t_0$.

On the other hand,

$$\begin{aligned} \langle W \rangle_t - W(t) &= \frac{1}{t} \int_0^t W(s) ds - W(t) \\ &\leq \frac{1}{t} \int_0^t \left(V_\infty + \gamma e^{-B_\infty s} - \frac{\gamma^{p+1}A_-}{s^{p+d}} \chi_{s \geq t_0+1} \right) ds - \left(V_\infty + \gamma e^{-B_0 t} - \frac{\gamma^{p+1}A_+}{\langle t \rangle^{p+d}} \right) \\ &\leq \frac{1}{t} \int_0^t \gamma e^{-B_\infty s} ds + \frac{\gamma^{p+1}A_+}{\langle t \rangle^{p+d}} \leq \frac{C(\gamma + \gamma^{p+1}A_+)}{1+t}. \end{aligned}$$

□

Corollary 4.1. *For small enough γ , we have*

- (i) $\langle W \rangle_t > W(t)$ for all t .
- (ii) $\langle W \rangle_t$ is a decreasing function.
- (iii) $\langle W \rangle_t > \langle W \rangle_{s,t}$, $\forall s \in (0, t)$.

Proof. Same as Corollary 3.1.

□

For the right side estimate, we estimate $\inf_{s < t} \langle W \rangle_{s,t}$

$$\begin{aligned} \langle W \rangle_{s,t} &\geq V_\infty - \gamma \langle g \rangle_{s,t} \geq V_\infty + \frac{\gamma}{t-s} \int_s^t -\frac{\gamma^p A_+}{\langle r \rangle^{p+d}} dr \\ &\geq V_\infty - \gamma^{p+1} A_+ \sup_{s < t} \left(\frac{1}{t-s} \int_s^t \frac{1}{\langle r \rangle^{p+d}} dr \right) \\ &\geq V_\infty - \frac{C\gamma^{p+1}A_+}{\langle t \rangle}, \end{aligned} \tag{4.4}$$

where $C = \int_0^\infty \frac{1}{\langle r \rangle^{p+d}} dr$ since $p+d > 1$. with the same method, we know that

$$\begin{aligned} \langle W \rangle_{s,t} &\leq V_\infty - \gamma \langle h \rangle_{s,t} \\ &\leq V_\infty + \frac{\gamma}{t-s} \int_s^t \left(e^{-B_\infty r} - \frac{\gamma^p A_-}{r^{p+d}} \chi_{\{r \geq t_0+1\}} \right) dr \\ &\leq V_\infty + \gamma. \end{aligned} \tag{4.5}$$

Examining Definition 2, we also notice that

$$V_\infty - \frac{\gamma^{p+1}A_+}{\langle t \rangle} \leq W(t) \leq V_\infty + \gamma.$$

4.3. The Left Side.

Lemma 4.3. *Let $W \in \mathcal{W}$ and let K and a_0 satisfy the Assumptions A1-A4. If k and a_0 satisfy the Reversal Criterion, then for all sufficiently small γ we have the inequalities*

$$c \frac{\gamma^{p+1}}{t^{d+p}} \chi \{t \geq t_0\} \leq r_W^L(t) \leq \frac{C(\gamma + \gamma^{p+1} A_+)^{p+1}}{(1+t)^{p+d}}.$$

Proof. We will first prove a careful upper bound of $f_+(t, \mathbf{x}; \mathbf{v})$. Recall the boundary condition on the left of the cylinder

$$f_+(t, \mathbf{x}; \mathbf{v}) = \int_{u_x \geq W(t)} K(\mathbf{v} - \mathbf{i}W(t); \mathbf{u} - \mathbf{i}W(t)) f_-(t, \mathbf{x}; \mathbf{u}) d\mathbf{u}.$$

and the precollision characteristic functions

$$\begin{aligned} \chi_0(t, \mathbf{u}) &= \chi \left\{ \mathbf{u} : \forall s \in (0, t), \text{ either } u_x \neq \langle W \rangle_{s,t} \text{ or } |u_\perp| > \frac{2r}{t-s} \right\}, \\ \chi_1(t, \mathbf{u}) &= \chi \left\{ \mathbf{u} : \exists s \in (0, t) \text{ s.t. } u_x = \langle W \rangle_{s,t} \text{ and } |u_\perp| \leq \frac{2r}{t-s} \right\}. \end{aligned}$$

We again write

$$f_-(t, \mathbf{x}; \mathbf{u}) = f_+(\tau, \boldsymbol{\xi}; \mathbf{u}) \chi_1(t, \mathbf{u}) + f_0(\mathbf{u}) \chi_0(t, \mathbf{u}),$$

which gives

$$f_+(t, \mathbf{x}; \mathbf{v}) = \int_{u_x \geq W(t)} K(\mathbf{v} - \mathbf{i}W(t); \mathbf{u} - \mathbf{i}W(t)) [f_+(\tau, \boldsymbol{\xi}; \mathbf{u}) \chi_1(t, \mathbf{u}) + f_0(\mathbf{u}) \chi_0(t, \mathbf{u})] d\mathbf{u}.$$

That is,

$$\begin{aligned} a_+(t, \mathbf{x}; v_x) b(v_\perp) &= b(v_\perp) \int_{u_x \geq W(t)} k(v_x - W(t), u_x - W(t)) \\ &\quad \times \{a_+(\tau, \boldsymbol{\xi}; u_x) b(u_\perp) \chi_1(t, \mathbf{u}) + a_0(u_x) b(u_\perp) \chi_0(t, \mathbf{u})\} d\mathbf{u}. \end{aligned}$$

We then notice by Lemma 4.1 (iii) that

$$\begin{aligned} a_+(t, \mathbf{x}; v_x) b(v_\perp) &\leq b(v_\perp) a_+^* \int_{W(t)}^{\langle W \rangle_t} k(v_x - W(t), u_x - W(t)) du_x \\ &\quad + b(v_\perp) \int_{u_x \geq W(t)} k(v_x - W(t), u_x - W(t)) a_0(u_x) du_x \\ &\leq b(v_\perp) C\gamma a_+^* + b(v_\perp) \int_{u_x \geq W(t)} k(v_x - W(t), u_x - W(t)) a_0(u_x) du_x \end{aligned}$$

by A2 and Lemma 4.2, where a_+^* is defined as in (3.13). Now recall the Reversal Criterion

$$\int_{u_x \geq V_\infty} k(0, u_x - V_\infty) a_0(u_x) du_x < a_0(V_\infty).$$

Hence by A4 and continuity, $\exists \delta > 0$ such that

$$\sup_{\substack{t, x \in \partial\Omega(t) \\ v_x \in [V_\infty - 2\gamma, V_\infty + 2\gamma]}} \int_{u_x \geq W(t)} k(v_x - W(t), u_x - W(t)) a_0(u_x) du_x \leq a_0(V_\infty) - \delta$$

for all γ small enough. We thus arrive at

$$a_+(t, \mathbf{x}; v_x) b(v_\perp) \leq b(v_\perp) C\gamma a_+^* + b(v_\perp) (a_0(V_\infty) - \delta).$$

That is,

$$b(v_\perp) a_+^* \leq \frac{1}{1 - C\gamma} (a_0(V_\infty) - \delta) b(v_\perp) \leq b(v_\perp) \left(a_0(V_\infty) - \frac{\delta}{2} \right) \quad (4.6)$$

for small γ . Now recall that

$$\begin{aligned} r_W^L(t) &= \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{u_x \geq W(t)} d\mathbf{u} \ell(u_x - W(t)) \{f_0(\mathbf{u}) - f_-(t, \mathbf{x}, \mathbf{u})\} \\ &= \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{u_x \geq W(t)} d\mathbf{u} \ell(u_x - W(t)) [f_0(\mathbf{u}) - f_+(\tau, \boldsymbol{\xi}, \mathbf{u}) \chi_1(t, \mathbf{u}) - f_0(\mathbf{u}) \chi_0(t, \mathbf{u})] \\ &= \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} \int_{W(t)}^{\langle W \rangle_t} du_x \ell(u_x - W(t)) (f_0(\mathbf{u}) - f_+(\tau, \boldsymbol{\xi}, \mathbf{u})). \end{aligned}$$

With (4.6) we obtain

$$r_W^L(t) \geq \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} b(u_{\perp}) du_{\perp} \int_{W(t)}^{\langle W \rangle_t} du_x \ell(u_x - W(t)) (a_0(u_x) - a_0(V_{\infty}) + \frac{\delta}{2}).$$

For small enough γ , we have by continuity of a_0 that

$$a_0(u_x) - a_0(V_{\infty}) + \frac{\delta}{2} \geq \frac{\delta}{4} > 0,$$

which implies

$$\begin{aligned} r_W^L(t) &\geq C \frac{\delta}{4} \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} b(u_{\perp}) du_{\perp} \int_{W(t)}^{\langle W \rangle_t} du_x \ell(u_x - W(t)) \\ &\geq C \frac{\delta}{4} \frac{(\langle W \rangle_t - W(t))^{p+1}}{(1+t)^{d-1}} \end{aligned}$$

Recalling Lemma 4.2, we have

$$\langle W \rangle_t - W(t) \geq \frac{C\gamma}{t} \chi\{t \geq t_0\},$$

whence

$$r_W^L(t) \geq C \frac{\gamma^{p+1}}{t^{d+p}} \chi\{t \geq t_0\}$$

for small enough γ . This is the desired lower bound of r_W^L .

For the upper bound of r_W^L , we have

$$\begin{aligned} |r_W^L(t)| &= \left| \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} \int_{W(t)}^{\langle W \rangle_t} du_x \ell(u_x - W(t)) (f_0(\mathbf{u}) - f_+(\tau, \boldsymbol{\xi}, \mathbf{u})) \right| \\ &\leq \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} \int_{W(t)}^{\langle W \rangle_t} du_x \ell(u_x - W(t)) |f_0(\mathbf{u}) - f_+(\tau, \boldsymbol{\xi}, \mathbf{u})| \\ &\leq C \int_{\partial\Omega_L(t)} dS_{\mathbf{x}} \int_{W(t)}^{\langle W \rangle_t} du_x \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}} du_{\perp} \ell(u_x - W(t)) b(u_{\perp}). \end{aligned}$$

Splitting the integral according to the regions $\tau < t/2$ and $\tau \geq t/2$, we have

$$\begin{aligned} r_W^L(t) &\leq \frac{C(\langle W \rangle_t - W(t))^{p+1}}{(1+t)^{d-1}} + C \int_{W(t)}^{\langle W \rangle_t} du_x \int_{|u_{\perp}| \leq \frac{2r}{t-\tau}, \tau \geq \frac{t}{2}} du_{\perp} \ell(u_x - W(t)) b(u_{\perp}) \\ &= I + II. \end{aligned}$$

By Assumption A3 and Lemma 4.2,

$$I \leq \frac{C \left(\frac{C(\gamma + \gamma^{p+1} A_+)}{1+t} \right)^{p+1}}{(1+t)^{d-1}} \leq C \frac{(\gamma + \gamma^{p+1} A_+)^{p+1}}{(1+t)^{d+p}}.$$

For the second term, by the precollision condition (3.7), we notice that

$$\begin{aligned} u_x &= \langle W \rangle_{\tau, t} \leq V_{\infty} - \gamma \langle h \rangle_{\tau, t} \\ &\leq V_{\infty} + \sup_{\frac{t}{2} \leq \tau \leq t} \frac{\gamma}{t-\tau} \int_{\tau}^t \left(e^{-B_{\infty} r} - \frac{\gamma^p A_-}{r^{p+d}} \chi\{r \geq t_0 + 1\} \right) dr \\ &\leq V_{\infty} + C\gamma e^{-B_{\infty} \frac{t}{2}}. \end{aligned}$$

By Assumption A3 again, this inequality allows us to estimate the second term as

$$\begin{aligned}
II &\leq C \int_{W(t)}^{V_\infty + C\gamma e^{-B_\infty \frac{t}{2}}} du_x \int_{|u_\perp| \leq \frac{2v}{t-\tau}, \tau \geq \frac{t}{2}} du_\perp \ell(u_x - W(t)) b(u_\perp) \\
&\leq C \int_0^{V_\infty - W(t) + C\gamma e^{-B_\infty \frac{t}{2}}} dz |z|^p \\
&\leq C \left(V_\infty - W(t) + C\gamma e^{-B_\infty \frac{t}{2}} \right)^{p+1}.
\end{aligned}$$

By definition of W , we have

$$V_\infty - W(t) \leq -\gamma e^{-B_0 t} + \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}} \leq \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}},$$

so that

$$II \leq \left(\frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}} + C\gamma e^{-B_\infty \frac{t}{2}} \right)^{p+1} \leq C \left(\frac{C\gamma + \gamma^{p+1} A_+}{\langle t \rangle^{p+d}} \right)^{p+1}.$$

Putting I and II together, we have

$$r_W^L(t) \leq C \frac{(\gamma + \gamma^{p+1} A_+)^{p+1}}{(1+t)^{d+p}},$$

which is the claimed upper bound for $r_W^L(t)$. \square

4.4. The Right Side.

Lemma 4.4. *Under the same assumptions as in Lemma 4.3, for γ small enough, there is a $\varepsilon > 0$ such that*

$$|r_W^R(t)| \leq \frac{C\gamma^{(1+\varepsilon)(p+1)} A_+^{p+1}}{t^{p+d}} \chi\{t \geq t_0\}.$$

Proof. As in the begining of the proof of Lemma 3.4, we have $r_W^R(t) = 0$ for all $t \leq t_0$ because W is decreasing. Setting $t \geq t_0$ and recalling the boundary condition on the right side of the cylinder, we have

$$f_+(t, \mathbf{x}; \mathbf{v}) = \int_{u_x \leq W(t)} K(\mathbf{v} - \mathbf{i}W(t); \mathbf{u} - \mathbf{i}W(t)) f_-(t, \mathbf{x}; \mathbf{u}) d\mathbf{u}.$$

Writing

$$f_-(t, \mathbf{x}; \mathbf{u}) = f_+(\tau, \boldsymbol{\xi}; \mathbf{u}) \chi_1(t, \mathbf{u}) + f_0(\mathbf{u}) \chi_0(t, \mathbf{u}),$$

we have

$$\begin{aligned}
f_+(t, \mathbf{x}; \mathbf{v}) &= b(v_\perp) \int_{u_x \leq W(t)} k(v_x - W(t), u_x - W(t)) \\
&\quad \times \{a_+(\tau, \boldsymbol{\xi}; u_x) b(u_\perp) \chi_1(t, \mathbf{u}) + a_0(u_x) b(u_\perp) \chi_0(t, \mathbf{u})\} d\mathbf{u}.
\end{aligned}$$

We then estimate

$$\begin{aligned}
f_+(t, \mathbf{x}; \mathbf{v}) &\leq b(v_\perp) \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} k(v_x - W(t), u_x - W(t)) a_+(\tau, \boldsymbol{\xi}; u_x) du_x \\
&\quad + b(v_\perp) \int_{-\infty}^{W(t)} k(v_x - W(t), u_x - W(t)) a_0(u_x) du_x
\end{aligned}$$

The last integral is uniformly bounded due to Assumption A4. For the other integral we need an estimate on the size of $W(t) - \inf_{s < t} \langle W \rangle_{s,t}$. With (4.4) and

$$W(t) \leq V_\infty - \gamma h(t) \leq V_\infty + \gamma e^{-B_\infty t},$$

we get

$$W(t) - \inf_{s < t} \langle W \rangle_{s,t} \leq \gamma e^{-B_\infty t} + \frac{C\gamma^{p+1} A_+}{\langle t \rangle}. \quad (4.7)$$

Thus we can estimate

$$b(v_\perp) \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} k(v_x - W(t), u_x - W(t)) a_+(\tau, \boldsymbol{\xi}; u_x) du_x \leq b(v_\perp) C\gamma a_+^*,$$

where a_+^* is as in (3.13), so that we have $b(v_\perp)a_+^* \leq b(v_\perp)C\gamma a_+^* + Cb(v_\perp)$. That is,

$$b(v_\perp)a_+^* \leq Cb(v_\perp)$$

for small enough γ . With this upper bound of $f_+(\tau, \xi, u)$ we arrive at

$$\begin{aligned} |r_W^R(t)| &= \left| \int_{\partial\Omega_R(t)} dS_x \int_{u_x \leq W(t)} d\mathbf{u} \ell(u_x - W(t)) \{f_-(t, \mathbf{x}; \mathbf{u}) - f_0(\mathbf{u})\} \right| \\ &= \left| \int_{\partial\Omega_R(t)} dS_x \int_{u_x \leq W(t)} d\mathbf{u} \ell(u_x - W(t)) \{f_+(\tau, \xi; \mathbf{u})\chi_1(t, \mathbf{u}) + f_0(\mathbf{u})\chi_0(t, \mathbf{u}) - f_0(\mathbf{u})\} \right| \\ &= \left| \int_{\partial\Omega_R(t)} dS_x \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} d\mathbf{u} \ell(u_x - W(t)) \{f_+(\tau, \xi; \mathbf{u}) - f_0(\mathbf{u})\} \right| \\ &\leq \int_{\partial\Omega_R(t)} dS_x \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} d\mathbf{u} \ell(u_x - W(t)) (f_+(\tau, \xi; \mathbf{u}) + f_0(\mathbf{u})) \\ &\leq C \int_{|u_\perp| \leq \frac{2r}{t-\tau}} du_\perp \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} du_x \ell(u_x - W(t)) b(u_\perp). \end{aligned}$$

Split the integral at $\tau = t/2$. As distinguished from the irreversible case, the $\tau < \frac{t}{2}$ part provides the dominant contribution this time. We have

$$I = C \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} du_x \int_{|u_\perp| \leq \frac{2r}{t-\tau}, \tau < \frac{t}{2}} du_\perp \ell(u_x - W(t)) b(u_\perp) \leq C \frac{(W(t) - \inf_{s < t} \langle W \rangle_{s,t})^{p+1}}{(1+t)^{d-1}}$$

Plugging (4.7) into this expression, we have:

$$I \leq \frac{C}{(1+t)^{d-1}} \left(\gamma e^{-B_\infty t} + \frac{C\gamma^{p+1}A_+}{\langle t \rangle} \right)^{p+1}$$

for $t \geq t_0$.

On the other hand, for the $\tau \in [\frac{t}{2}, t]$ part, we know that

$$\begin{aligned} u_x &= \langle W \rangle_{\tau,t} \geq V_\infty - \gamma \langle g \rangle_{\tau,t} \\ &\geq V_\infty + \inf_{\frac{t}{2} \leq \tau \leq t} \frac{\gamma}{t-\tau} \int_\tau^t \left(e^{-B_0 r} - \frac{\gamma^p A_+}{\langle r \rangle^{p+d}} \right) dr \\ &\geq V_\infty + \inf_{\frac{t}{2} \leq \tau \leq t} \frac{\gamma}{t-\tau} \int_\tau^t \left(-\frac{\gamma^p A_+}{\langle r \rangle^{p+d}} \right) dr \geq V_\infty - C \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}}, \end{aligned}$$

which yields

$$\begin{aligned} II &= C \int_{\inf_{s < t} \langle W \rangle_{s,t}}^{W(t)} du_x \int_{|u_\perp| \leq \frac{2r}{t-\tau}, \frac{t}{2} \leq \tau \leq t} du_\perp \ell(u_x - W(t)) b(u_\perp) \\ &\leq C \int_{V_\infty - C \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}}}^{W(t)} \ell(u_x - W(t)) du_x \\ &\leq C \left| W(t) - V_\infty - C \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}} \right|^{p+1}. \end{aligned}$$

By (4.1), we estimate

$$\begin{aligned} &\left| W(t) - V_\infty - C \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}} \right| \\ &\leq \max \{ \gamma e^{-B_\infty t}, \gamma e^{-B_0 t} \} + \max \left\{ \frac{\gamma^{p+1} A_+}{t^{p+d}} \chi \{t \geq t_0 + 1\}, \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}} \right\} + C \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}} \\ &\leq C \left(\gamma e^{-B_\infty t} + \frac{\gamma^{p+1} A_+}{\langle t \rangle^{p+d}} \right). \end{aligned}$$

where in the last line we used the fact that $B_\infty \leq B_0$ and $A_- \leq A_+$. Indeed, for large t , we have

$$-g(t, \gamma) \sim -\frac{\gamma^p A_+}{\langle t \rangle^{p+d}} \text{ and } -h(t, \gamma) \sim -\frac{\gamma^p A_-}{\langle t \rangle^{p+d}}$$

so that we require $A_+ \geq A_-$ to make sure that $V_\infty - \gamma g(t, \gamma) \leq V_\infty - \gamma h(t, \gamma)$.

With the estimates for I and II in hand, we have

$$\begin{aligned} |r_W^R(t)| &\leq \frac{C}{(1+t)^{d-1}} \left(\gamma e^{-B_\infty t} + \frac{C\gamma^{p+1}A_+}{\langle t \rangle} \right)^{p+1} \chi\{t \geq t_0\} \\ &\quad + C \left(\gamma e^{-B_\infty t} + \frac{\gamma^{p+1}A_+}{\langle t \rangle^{p+d}} \right)^{p+1} \chi\{t \geq t_0\}. \end{aligned} \quad (4.8)$$

Because on its face estimate (4.8) alone is not enough to deduce for the reversal case that

$$r_W^L(t) + r_W^R(t) \geq 0,$$

we now give a more precise bound. Let us recall from Definition 2 that $t_0 = K_0 |\log \gamma|$ with $\frac{1}{B_0} \leq K_0 \leq \frac{2}{B_0}$. The key here is to choose γ small enough so that the quantity $B_\infty K_0$ can be bounded below by

$$B_\infty K_0 \geq \frac{B_\infty}{B_0} = \frac{\min_{V \in [V_\infty - \gamma, V_\infty + \gamma]} F'_0(V)}{\max_{V \in [V_\infty - \gamma, V_\infty + \gamma]} F'_0(V)} \geq 1 - \varepsilon, \varepsilon > 0$$

which can be made close to 1. For $t \geq t_0$, we write

$$\begin{aligned} e^{-B_\infty t} &= e^{-(1-\varepsilon)B_\infty t} e^{-\varepsilon B_\infty t} \leq e^{-(1-\varepsilon)B_\infty t} e^{-\varepsilon B_\infty t_0} \\ &\leq e^{-(1-\varepsilon)B_\infty t} e^{\varepsilon B_\infty K_0 \log \gamma} = e^{-(1-\varepsilon)B_\infty t} \gamma^{\varepsilon B_\infty K_0} \leq e^{-(1-\varepsilon)B_\infty t} \gamma^{\frac{\varepsilon}{2}}, \end{aligned}$$

because $\gamma < 1$. By the choice $\varepsilon \in (0, 2p)$ and because $A_+ > 0$, estimate (4.8) becomes

$$\begin{aligned} |r_W^R(t)| &\leq \frac{C}{(1+t)^{d-1}} \left(\gamma e^{-(1-\varepsilon)B_\infty t} \gamma^{\frac{\varepsilon}{2}} + \frac{C\gamma^{p+1}A_+}{\langle t \rangle} \right)^{p+1} \chi\{t \geq t_0\} \\ &\quad + C \left(\gamma e^{-(1-\varepsilon)B_\infty t} \gamma^{\frac{\varepsilon}{2}} + \frac{\gamma^{p+1}A_+}{\langle t \rangle^{p+d}} \right)^{p+1} \chi\{t \geq t_0\} \\ &\leq \frac{C\gamma^{(1+\frac{\varepsilon}{2})(p+1)}A_+^{p+1}}{\langle t \rangle^{p+d}} \chi\{t \geq t_0\} + \frac{C\gamma^{(1+\frac{\varepsilon}{2})(p+1)}A_+^{p+1}}{\langle t \rangle^{(p+d)(p+1)}} \chi\{t \geq t_0\}. \end{aligned}$$

Since $p > 0$, the second term is absorbed into the first term for large enough t_0 , so that

$$|r_W^R(t)| \leq \frac{C\gamma^{(1+\frac{\varepsilon}{2})(p+1)}A_+^{p+1}}{\langle t \rangle^{p+d}} \chi\{t \geq t_0\}$$

as claimed. \square

Putting together Lemmas 4.3 and 4.4, we conclude that

$$R_W(t) \geq \left(\frac{c_1\gamma^{p+1}}{t^{p+d}} - \frac{C_2\gamma^{(1+\varepsilon)(p+1)}A_+^{p+1}}{t^{p+d}} \right) \chi\{t \geq t_0\}$$

and

$$R_W(t) \leq \frac{C_1(\gamma + \gamma^{p+1}A_+)^{p+1}}{(1+t)^{p+d}} + \frac{C_2\gamma^{(1+\varepsilon)(p+1)}A_+^{p+1}}{t^{p+d}} \chi\{t \geq t_0\}.$$

Theorem 4.1 then follows since γ is small.

REFERENCES

- [1] K. Aoki, G. Cavallaro, C. Marchioro and M. Pulvirenti, *On the motion of a body in thermal equilibrium immersed in a perfect gas*, ESAIM: Mathematical Modelling and Numerical Analysis **42** (2008), 263-275.
- [2] K. Aoki, T. Tsuji and G. Cavallaro, *Approach to steady motion of a plate moving in a free-molecular gas under a constant external force*, Phys. Rev. E **80** (2009), 016309.
- [3] S. Caprino, G. Cavallaro, and C. Marchioro, *On a microscopic model of viscous friction*, Math. Models Meth. Appl. Sci. **17** (2007), 1369-1403.
- [4] S. Caprino, C. Marchioro, and M. Pulvirenti, *Approach to equilibrium in a microscopic model of friction*, Commun. Math. Phys. **264** (2006), 167-189.
- [5] G. Cavallaro, *On the motion of a convex body interacting with a perfect gas in the mean-field approximation*, Rend. Mat. Appl. **27** (2007), 123-145.
- [6] G. Cavallaro and C. Marchioro, *On the Motion of an Elastic Body in a Free Gas*, Reports on Mathematical Physics **69** (2012), 251-264.
- [7] X. Chen and W. Strauss, *Approach to Equilibrium of a Body Colliding Specularly and Diffusely with a Sea of Particles*, Arch. Rational Mech. Anal. **211** (2014), 879-910.
- [8] R. Glassey, *The Cauchy Problem in Kinetic Theory*, SIAM, Philadelphia, PA, 1996.
- [9] F. Sisti, C. Ricciuti, *Effects of concavity on the motion of a body immersed in a Vlasov gas*, 37pp, arXiv:1401.5942.
- [10] Y. Sone, *Kinetic Theory and Fluid Dynamics*, Birkhäuser, 2002.
- [11] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Springer, 1991.
- [12] T. Tsuji and K. Aoki, *Decay of an oscillating plate in a free-molecular gas*, in: Rarefied Gas Dynamics, D. A. Levin et al. eds., AIP, Melville, 2011, 140-145.

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